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General Topology and its Applications 8 (1978) 299–301.
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SCATTERED SPACES AND THEIR COMPACTIFICATIONS

R.C. SOLOMON
Department of Mathematics, University of Malawi, Zamba, Malawi, Africa

Received 4 February 1975

We show that a known restriction on the cardinalities of closures of subspaces of scattered spaces, $|\bar{A}| \leq 2^{|A|}$, cannot be improved to $|\bar{A}| \leq |A|^{\lambda}$, for any λ. We then find a wide class of T.3½. scattered spaces which have no scattered compactification: these spaces are derived from regular filters over cardinals bigger than $\aleph_1$.

A.M.S. Subj. Class.: 54A25, 54D35, 54G99
scattered spaces regular filters
compactifications

0.

A *scattered* space is one with no subspace that is dense in itself. Equivalently, every subspace has an isolated point. It follows from this that every subspace must have a *dense* set of isolated points.

One can arrange a scattered space in a transfinite hierarchy as follows:

$X_0 = X$,

$X_{\alpha+1} =$ the set of accumulation points of X_α,

$X_\delta = \bigcap_{\alpha<\delta} X_\alpha$ for δ a limit ordinal.

Each X_α is closed. If $X_\alpha \neq \emptyset$, $X_\alpha - X_{\alpha+1} \neq \emptyset$ also. So for some ordinal α, $X_\alpha = \emptyset$. The least α for which this is true is called the *level* of X, $l(X)$. For $x \in X$, the least α for which $x \notin X_\alpha$ is the level of x, $l(x)$.

If X is compact, then $l(X)$ is a successor ordinal α. Furthermore only finitely many points of X are of level α.

If X is T.3½., X has a Hausdorff compactification. This need not be scattered. For example, βN is a non-scattered compactification of the scattered space N. But if X is locally compact, the one-point compactification of X is scattered. Compact scattered spaces are 0-dimensional, i.e. have a base of clopen sets.

Throughout every space is assumed to be Hausdorff. [3] and [4] contain many of the basic results.

1.

If X is a compact scattered space and $A \subseteq X$, $|\bar{A}| \leq 2^{|A|}$. This was shown in [3]. They asked if this could be sharpened to $|\bar{A}| \leq |A|^{\lambda}$ for some fixed cardinal λ. We show that this is not possible.

Theorem 1. *For every cardinal λ, there is a compact scattered space X of power greater than μ^{λ}, but with a dense subspace of power μ.*

Proof. Let κ be the first cardinal so that $2^{\kappa} > 2^{\lambda}$. Then $\lambda < \kappa \leq 2^{\lambda} < 2^{\kappa}$. Define Y as follows:

The elements of Y are all the well-ordered sequences of 0's and 1's of order type less than or equal to κ.

If $a, b \in Y$, write $a \leq b$ if b is an extension of a. Basic neighbourhoods of b are sets of the form $\{a : c < a \leq b\}$ for each $c < b$. We have the following facts:

1) Y is Hausdorff.

2) $|Y| = 2^{\kappa}$. There are 2^{κ} points of order type κ.

3) Y is scattered. If $A \subseteq Y$, a point of A of least possible order type will be isolated in A.

4) The points in Y of order type less than κ are dense in Y. So Y has a dense subset of power $\kappa \cdot 2^{\lambda} = 2^{\lambda}$.

5) Y is locally compact. For if $b \in Y$, the neighbourhood $\{a : a \leq b\}$ is homeomorphic to the compact space $[0, \alpha]$, where α is the order type of b.

Hence Y has a one-point compactification X which will have all the properties of the theorem, with $\mu = 2^{\lambda}$.

2.

In [4] and again in [3] the question was raised:

Does every T.3½. scattered space have a scattered compactification?

This was answered by P. Nyikos in the negative [2]:

Let κ be an uncountable cardinal. Take the product of two-point discrete spaces. Let every point be isolated, except one which has the ordinary product topology neighbourhoods. Then this space is T.3½. and scattered, but has no scattered compactification.

Before we go on to discuss other examples, some further notation is necessary.

$X - X_1$ is a discrete space of power λ say. Henceforth we identify $X - X_1$ with λ. If $x \in X_1$, the set $\{\lambda \cap O : O$ a nbd of $x\}$, is a non-principal filter over λ. Conversely, if p is a non-principal filter over λ, $\{p\} \cup \lambda$ will denote the space in which every point of λ is isolated, and p has neighbourhoods of the form $\{p\} \cup A$ for each $A \in p$. We shall identify points $x \in X_1$ and the non-principal filters they generate.

A filter p is called κ-regular if it contains a subset $\mathscr{F}$ of power κ such that the intersection of infinitely many members of $\mathscr{F}$ is empty.

If p is κ-regular, and $A \in p$, then $|A| \geqslant \kappa$. Regular ultrafilters are very important in model-theory [1]. They give rise to *universal* ultrapowers. It has for long been an open question whether or not there exists a non-principal ultrafilter p, all of whose sets have power κ, which is not κ-regular.

Theorem 2. *If κ is a cardinal greater than $\aleph_1$, and p is κ-regular, then $\{p\} \cup \lambda$ has no scattered compactification. (p is a filter over λ, for some $\lambda \geqslant \kappa$.)*

Proof. Suppose the theorem first fails at κ. Suppose X is a scattered compactification of $\{p\} \cup \lambda$, of least possible level α. Then as X is 0-dimensional, p is contained in a clopen set which has no other points of level greater or equal to $l(p)$. So we can assume that p is of level α, and is the only point of level α. By the minimality of κ and α, we can then assume that if $x \in X - \{p\}$, x is at best $\aleph_1$-regular.

Let $\mathcal{F}$ be the subset of p that makes p κ-regular. For each $A \in \mathcal{F}$, there is a clopen neighbourhood O_A of p such that $(\lambda - A) \cap O_A = \emptyset$. Hence $O_A \cap \lambda \subseteq A$. The set $\mathcal{F}' = \{O_A \cap \lambda : A \in \mathcal{F}\}$ also makes p κ-regular, so we can assume without loss of generality that $\bar{A}$ is clopen for each $A \in \mathcal{F}$.

For $A \in \mathcal{F}$, let $\lambda_A = l(X - \bar{A})$. Let p_A be the (finite) set of points in $X - \bar{A}$ of level λ_A.

Now let $B \subseteq \mathcal{F}$ be of power $\aleph_1$. If $A \in B$, $x \in p_A$, x is at best $\aleph_1$-regular. So $x \in \bar{C}$ for at most $\aleph_1$ C's in $\mathcal{F}$. $|B| = \aleph_1$, so there is $A_1 \in \mathcal{F}$ so that $\bar{A}_1 \cap p_A = \emptyset$ for every $A \in B$. p_A consisted of the points of $X - \bar{A}$ of highest level, hence $l(\bar{A}_1 - \bar{A}) < \lambda_A$.

We can repeat this process to obtain $A_2 \in \mathcal{F}$, so that $l(\overline{A_1 \cap A_2} - \bar{A}) < l(\bar{A}_1 - \bar{A}) < \lambda_A$ for every $A \in B$.

This process can be repeated any finite number of times. Each time we lower the ordinal assigned to each $A \in B$. For each $A \in B$ there is a number n so that after n applications of the process λ_A has been lowered to 0. So there is a fixed n, and an uncountable subset B' of B, so that after n applications λ_A has been lowered to 0 for every $A \in B'$,

$$\text{i.e.} \quad l(\overline{A_1 \cap \cdots \cap A_n} - \bar{A}) = 0 \quad \text{for all } A \in B'.$$

But $l(X) = 0$ only if $X = \emptyset$. Hence $\bar{A} \supseteq \overline{A_1 \cap \cdots \cap A_n}$ for all $A \in B'$. So $\bigcap_{A \in B'} A \neq \emptyset$, contradicting the fact that p is κ-regular. This proves the theorem.

References

[1] H.J. Keisler, On Cardinalities of Ultraproducts, Bull. Amer. Math. Soc. 70 (1964) 644–647.

[2] P. Nyikos, Not every scattered space has a scattered compactification, Notices Amer. Math. Soc. 21 (1974) 570.

[3] S. Mrowka, M. Rajagopalan and T. Soundararajan, A characterization of compact scattered spaces through chain limits, in: Topo 72, Springer-Verlag Lecture Notes in Mathematics, Vol. 378, pp. 288–297.

[4] Z. Semadeni, Sur les ensembles clairsemes, Rozpr. Matem. 19 (1959) 1–39.